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An assessment of systematic errors in beam tests on brittle materials

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In deriving the unit volume strength of brittle materials from beam bend test data, the shear and compressive stresses in the beam specimens are usually ignored. Depending on the span-to-depth ratio of the beam, these omissions may give rise to significant errors in the unit volume strength value, and in component failure probabilities derived from this strength value. In this paper, the effects of these systematic errors are considered for the 3-point beam as a function of span-to-depth ratio and for different values of the Weibull modulus. The relative errors are found to be small and conservative provided a sufficiently large span-to-depth ratio is used and the compressive/tensile strength ratio of the material is relatively high.

1. Introduction

The following expression, based on the Weibull probability distribution function [1], has been developed [2] for the prediction of the failure probability of a brittle component or specimen subjected to mechanical or thermal loads:

$$P_f = 1 - \exp\left[-\left(\frac{1}{m}!\right)^m \left(\frac{\sigma_{nom}}{\bar{\sigma}_{fv}}\right)^m \frac{V}{v}\,\Sigma(V)\right] \quad (1)$$

where P_f is the failure probability; m is the Weibull modulus; $(1/m!)$ is the gamma function of $(1/m + 1)$; σ_{nom} is a convenient nominal stress proportional to the load; $\bar{\sigma}_{fv}$ is the mean tensile fracture stress of unit volume of the material, i.e. the unit volume strength; V is the volume of the component (subscript c) or specimen (subscript s); and v is unit volume.

The quantity $\Sigma(V)$ is the "stress–volume integral" defined by the expression

$$\Sigma(V) = \int_{Vol}\left[\left(\frac{\sigma_1}{H(\sigma_1)\,\sigma_{nom}}\right)^m + \left(\frac{\sigma_2}{H(\sigma_2)\,\sigma_{nom}}\right)^m + \left(\frac{\sigma_3}{H(\sigma_3)\,\sigma_{nom}}\right)^m\right]\frac{dV}{V} \quad (2)$$

in which σ_1, σ_2 and σ_3 are the three principal stresses acting on the volume element dV, and $H(\sigma)$ is a step function such that

when $\sigma \geqslant 0$ $\quad H(\sigma) = 1$

when $\sigma < 0$ $\quad H(\sigma) = -\eta$

where η is the numerical ratio of the compressive to the tensile strength of the material.

The detailed development of Equation 1 is critically discussed in [2, 3], and [4] deals in detail with the unit volume strength concept. A typical application of the equation in a major design study is described in [5].

The material characteristics in Equation 1 are the quantities m, which is an inverse measure of the strength variability of a batch of nominally identical specimens, and $\bar{\sigma}_{fv}$, a strength parameter. Clearly these quantities must be determined before quantitative P_f calculations can be made in a particular case. (An alternative formulation of Equation 1 is avilable [2] in which the material strength is defined in terms of $\bar{\sigma}_{fa}$, the mean tensile fracture stress of unit surface area of the material, and the stress integral is taken over the surface area of the body instead of over the

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volume as in Equation 2. In the work presented in this paper the unit volume strength $\bar{\sigma}_{fv}$ is used as the strength term throughout. The relative merits of $\bar{\sigma}_{fv}$ and $\bar{\sigma}_{fa}$ are discussed in [4].)

The Weibull modulus and unit volume strength are readily obtained from fracture test data for a large batch of nominally identical specimens, loaded in such a way as to give a known stress distribution. The simple uniaxial tensile test specimen is not satisfactory for this purpose, and wide use is made of the simply supported beam specimen subjected to 3-point or 4-point bending [6]. The Weibull modulus m is usually derived graphically or by curve-fitting from the ranked fracture loads (or nominal stresses) and is not subjected to the errors discussed below. In deriving the unit volume strength, $\bar{\sigma}_{fv}$, the stress distribution in the beam specimen is required for calculation of $\Sigma(V_s)$, and it is customary [4, 7] to ignore the shear stresses and compressive stresses which occur in the specimens (η the compressive/tensile strength ratio is usually considerably greater than unity, typically 8 to 10). Systematic errors occur therefore in the calculated unit volume strength and in any failure probability values derived from it. An assessment of these errors is given in the present paper.

2. Derivation of unit volume strength

The unit volume strength is related to the mean value, $\bar{\sigma}_{nom}$, of the nominal fracture stress of a large batch of nominally identical specimens of the material and the stress–volume integral of the specimen by the expression [4]

$$\bar{\sigma}_{fv} = \bar{\sigma}_{nom}\left[\frac{V_s\Sigma(V_s)}{v}\right]^{1/m} \tag{3}$$

The derivation of Equation 3 from Equation 1 is detailed in [4]; it is convenient to use Equation 3 for the determination of $\bar{\sigma}_{fv}$ from the test results.

Since the errors to be considered are greatest in the case of the 3-point beam (see Fig. 1), this specimen is treated in detail. The nominal stress in beam work is invariably chosen as the maximum tensile stress due to bending, i.e. $3Wl/2bd^2$ for the 3-point beam (see Fig. 1a for symbols and Fig. 1c for the bending moment distribution). The mean nominal stress at fracture follows therefore directly from the mean fracture load. In obtaining the specimen volume ($V_s = b \times d \times l$), the portions of the specimen outside the supports (i.e. the "overhangs") are ignored. Expressions for the stress–volume integral are derived below firstly, as is usual, with shear stresses and compressive stresses ignored and then with due allowance for these stresses.

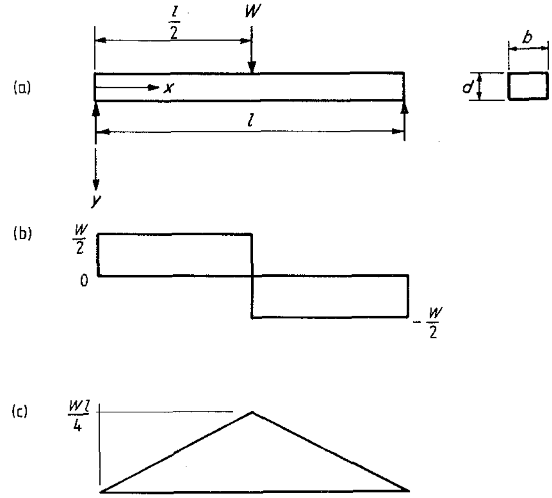

Figure 1 The 3-point bend specimen (a) notation, (b) shear force distribution, (c) bending moment distribution.

2.1. Approximate stress–volume integral

The direct stresses due to bending at a general point in the beam are

$$x < l/2, \qquad \sigma_x = 6Wxy/bd^3$$
$$x > l/2, \qquad \sigma_x = 6W(l-x)y/bd^3 \tag{4}$$

(The stresses in the y-direction and those normal to the plane of Fig. 1a are assumed zero throughout.) If shear stresses are neglected, then the above become the principal stresses and can be substituted directly into Equation 2. If, further, the compressive stresses are ignored (i.e. the integration is limited to $0 \leqslant y \leqslant d/2$) then, using the above nominal stress, the stress–volume integral (Equation 2) is obtained by simple integration as

$$\Sigma^*(V_s) = \frac{1}{2(m+1)^2} \tag{5}$$

(The asterisk is used to indicate that this is an approximate form for the integral.)

2.2. Exact stress–volume integral

The 3-point beam experiences the transverse shear force system depicted in Fig. 1b. The corresponding shear stress at a general point is given by [8]

$$x < l/2, \qquad \tau_{xy} = \frac{3W}{bd^3}\left(\frac{d^2}{4} - y^2\right)$$

$$x > l/2, \qquad \tau_{xy} = -\frac{3W}{bd^3}\left(\frac{d^2}{4} - y^2\right) \quad (6)$$

In the combined system of direct stresses (Equation 4) and shear stresses (Equation 6), the principal stresses σ_1 and σ_2 become

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \left(\frac{\sigma_x^2}{4} + \tau_{xy}^2\right)^{1/2} \quad (7)$$

Substituting for σ_x and τ_{xy} from Equations 4 and 6 respectively, these principal stresses and the previous nominal stress may be used in Equation 2 to give an "exact" expression for the stress–volume integral of the form

$$\Sigma(V_s) = \frac{2}{ld}\int_0^{l/2}\int_{-d/2}^{+d/2}\left(\frac{2}{ldH(\sigma_{1,2})}\right.$$

$$\left.\times\left\{xy \pm \left[x^2y^2 + \left(\frac{d^2}{4} - y^2\right)^2\right]^{1/2}\right\}\right)^m \mathrm{d}y\,\mathrm{d}x \quad (8)$$

(The integrand in Equation 8 consists of two terms which are identical except that one contains the $H(\sigma_1)$ term and the plus sign of the $\pm$ alternative, the other contains the $H(\sigma_2)$ term and the minus sign of the $\pm$ alternative.)

It will be noted that the integration limits for y ($+d/2$ to $-d/2$) are such that compressive stresses as well as shear stresses are taken into account and the limits for x (0 to $l/2$) exploit the symmetry of the stress distributions about the mid-span of the beam (Figs. 1b and c).

Using Equation 3, the effects of using the approximate (Equation 5) instead of the exact form of the stress–volume integral (Equation 8) on the derived unit volume strength value and subsequent failure probability values have been assessed for different values of span-to-depth ratio, l/d, and Weibull modulus, m. The value of η, the compressive/tensile strength ratio, is important; a value of 8 (as used in previous work [9]) was taken initially.

3. Calculation procedure

Equation 8 cannot be integrated analytically and a numerical evaluation using the Gaussian integration procedure [10] was used. The number of Gauss points was increased until the difference between successive values of $\Sigma(V_s)$ in an interactive process was less than 0.2%; an end-total of 15×15 Gauss points was used. As a check on accuracy, the same calculation was performed for a 3-point beam with l/d of 10 and m of 20, neglecting shear and compressive stresses; the value of $\Sigma^*(V_s)$ obtained was within 0.1% of the value calculated from Equation 5.

4. Results

4.1. Error in stress-volume integral

The relative difference (i.e. $\Delta\Sigma(V_s)/\Sigma(V_s)$ where $\Delta\Sigma(V_s) = \Sigma^*(V_s) - \Sigma(V_s)$) between the approximate stress–volume integral $\Sigma^*(V_s)$ (Equation 5) and the numerically evaluated value $\Sigma(V_s)$ (Equation 8), is shown in Fig. 2a as a function of l/d ratio for m-values of 5, 10, 15 and 20. The relative error in the stress–volume integral caused by neglecting shear and compressive stresses in a 3-point beam specimen is always negative (i.e. $\Sigma^*(V_s)$ is less than $\Sigma(V_s)$) and decreases (numerically) with m. It is seen that for $m = 5$ and a length-to-depth ratio greater than 3.25 the error will be less than 5%. For l/d ratios greater than 8 the error is less than 1% for all m-values studied (Fig. 2a).

4.2. Error in unit volume strength

The resultant error in the unit volume strength $\bar{\sigma}_{fv}$ can be calculated from the errors in $\Sigma(V_s)$ obtained above.

Using the normal error analysis techniques [11], and ignoring errors in the other parameters, the relative error in $\bar{\sigma}_{fv}$ is obtained from Equation 3 as

$$\frac{\Delta\bar{\sigma}_{fv}}{\bar{\sigma}_{fv}} = \frac{1}{m}\left[\frac{\Delta\Sigma(V_s)}{\Sigma(V_s)}\right] \quad (9)$$

where the prefix Δ denotes the absolute error in a quantity. $(\Delta\bar{\sigma}_{fv}/\bar{\sigma}_{fv})$ is plotted against l/d in Fig. 2b for the four m-values. Again, the error is always negative and values of the unit volume strength derived from the approximate stress–volume integral are therefore conservative.

The error decreases with increasing m and is generally small; for example, beams with l/d ratios greater than 3.5 give a relative error in $\bar{\sigma}_{fv}$ of less than 1% for the range of m considered.

4.3. Error in component failure probability

Values of component failure probability calculated from Equation 1 using erroneous values of unit volume strength will be systematically in error. These errors can be estimated from an error analysis based on Equation 1.

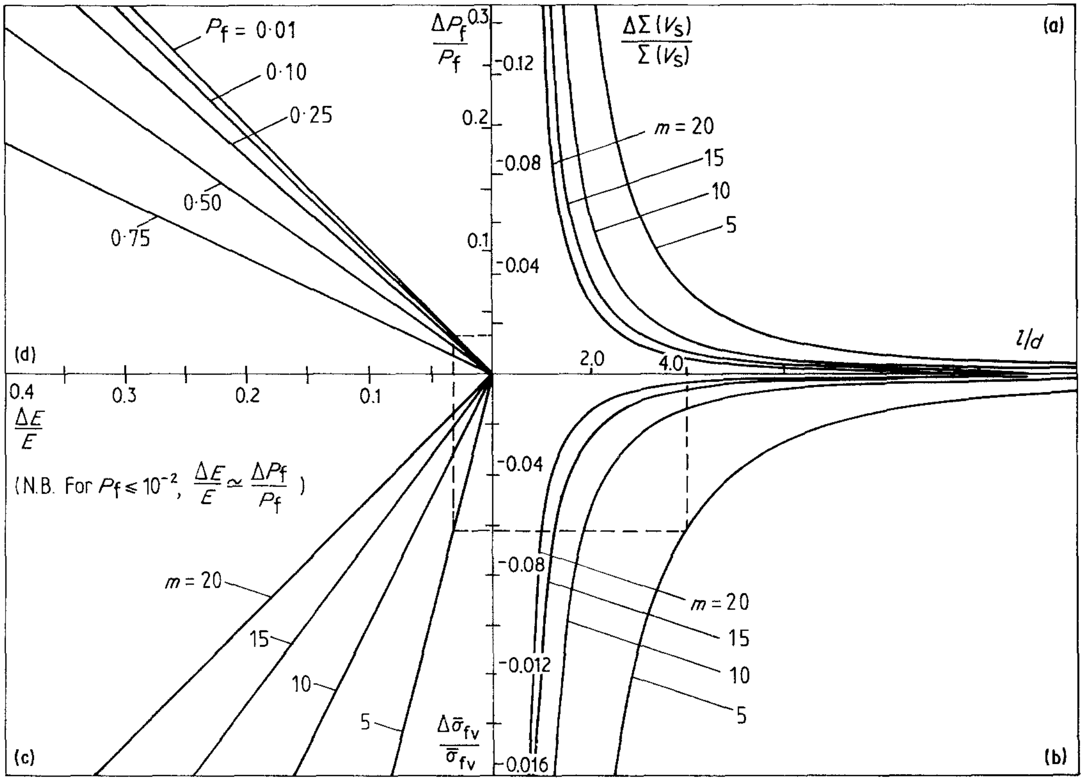

Figure 2 Nomograph for determining $\Delta\bar{\sigma}_{fv}/\bar{\sigma}_{fv}$ and $\Delta P_f/P_f$ for 3-point beam specimens of different l/d ratios and m = 5, 10, 15 and 20 (η = 8).

4.3.1. For small failure probabilities

Equation 1 may be rewritten as

$$\ln(1 - P_f) = E \tag{10}$$

where E is used to denote the quantity

$$\left[-\left(\frac{1}{m}!\right)^m \left(\frac{\sigma_{nom}}{\bar{\sigma}_{fv}}\right)^m \frac{V_c}{v} \sum(V_c)\right]$$

For values of component failure probability less than about 10^{-2}, it is satisfactory to write as an approximation, $P_f \approx -\ln(1 - P_f)$ or, from Equation 10,

$$P_f \approx -E \tag{11}$$

(The error in this approximation for a P_f value of 10^{-2} is 5×10^{-5}, i.e. about 0.5%.) For such cases, ignoring errors in quantities other than $\bar{\sigma}_{fv}$, it follows that the relative error in the probability of failure is given by

$$\frac{\Delta P_f}{P_f} \approx \frac{\Delta E}{E} = -m\left(\frac{\Delta\bar{\sigma}_{fv}}{\bar{\sigma}_{fv}}\right) \tag{12}$$

The quantity $(\Delta E/E)$ is plotted in Fig. 2c as a function of $(\Delta\bar{\sigma}_{fv}/\bar{\sigma}_{fv})$ for m-values of 5, 10, 15 and 20. For component failure probabilities less than 10^{-2} the abscissa in Fig. 2c can be taken as $\Delta P_f/P_f$ and the plot provides an acceptable approximation for the relative error. It is to be noted that for these small failure probabilities the relative error is independent of the actual value of failure probability and is generally positive.

4.3.2. For any failure probability

Where the failure probability exceeds 10^{-2} the approximation represented in Equations 11 and 12 may be unacceptable. In such cases Equation 1 must be treated without simplification, giving

$$\begin{aligned} \Delta P_f &= -\Delta E \exp E \\ &= -E \exp E \ \frac{\Delta E}{E} \\ &= -(1 - P_f)\ln(1 - P_f) \ \frac{\Delta E}{E} \end{aligned} \tag{13}$$

from which it follows that

$$\frac{\Delta P_f}{P_f} = -\frac{(1-P_f)\ln(1-P_f)}{P_f}\ \frac{\Delta E}{E}$$
$$= f(P_f)\ \frac{\Delta E}{E}\ . \quad (14)$$

It can be seen that, in general, the relative error in component failure probability depends on the value of the failure probability itself. The quantity $(\Delta P_f/P_f)$ is plotted in Fig. 2d against $(\Delta E/E)$ for several values of failure probability. It is to be noted that as the value of P_f tends to zero, $(\Delta P_f/P_f)$ tends towards the relative error for small failure probabilities (i.e. the limiting value of the function $f(P_f)$ in Equation 14 is 1).

Figs. 2a to d are presented in the form of a nomograph from which the relative error in unit volume strength for any given span-to-depth ratio of the test beam specimens, and the corresponding relative error in component failure probability, can be obtained. In the combined figure, an example is given in which results using test beams with an l/d ratio of 4 and an m-value of 5, are seen to give a relative error of 0.6% in $\bar{\sigma}_{fv}$ and 3% in P_f for a component which has a probability of failure of 0.01 or less.

5. Effect of compressive-to-tensile strength ratio

The above analysis was repeated for values of η, the ratio of the compressive-to-tensile strength, of 4, 2 and 1 in turn. The relative error in $\bar{\sigma}_{fv}$, as a result of neglecting shear and compressive stresses in the test specimens, and the corresponding relative error in the predicted failure probability of a component, were found to be small in all cases, except for $\eta = 1$. (This extreme case represents a material which has the same strength in compression as in tension and is included for comparison purposes.)

For the specific 3-point beam example cited above (i.e. $l/d = 4$, $m = 5$), relative errors in unit volume strength and component failure probability (for P_f 10^{-2}) are given in Table I for the four values of η.

TABLE I Relative errors in $\bar{\sigma}_{fv}$ and P_f for different values of η ($l/d = 4, m = 5, P_f \leqslant 10^{-2}$)

η	$\frac{\Delta\bar{\sigma}_{fv}}{\bar{\sigma}_{fv}}$ (%)	$\frac{\Delta P_f}{P_f}$ (%)
8	− 0.61	3.0
4	− 0.63	3.1
2	− 1.20	6.0
1	− 10.31	51.5

6. Conclusions

The effects of neglecting shear and compressive stresses in 3-point beam specimens used in brittle materials testing on the calculated unit volume strength are small for materials with a compressive to-tensile strength ratio significantly greater than unity, and are always conservative.

The magnitude of the error depends on the length-to-depth ratio of the beam specimens and the Weibull modulus of the materials. In the 3-point bend test a minimum length-to-depth ratio of 4 will ensure that the error will not exceed 1% for the range of Weibull modulus values normally encountered.

The resulting effects on the predicted probability of failure of components made from the material are also small, in general, and conservative. The relative error varies with the failure probability when this exceeds a value of $c.\ 10^{-2}$.

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